LECTURE 3: PRINCIPAL BUNDLES AND VECTOR BUNDLES

1. Introduction: Dirac's charge quantization

Before we start, let us summarize the main conclusion from last week's lecture about Maxwell theory:

- Maxwell theory can be conveniently rephrased in the language of differential forms by collecting the electric and magnetic fields into a 2-form F, called the *field strength*. Maxwell's equations can then be written as

$$dF = 0, \quad d \star F = \star J, \tag{1}$$

  where the 1-form J describes the distribution of charges and currents. The advantage of this formulation is that the equations (1) now *make sense on any manifold M equipped with a (pseudo)-riemannian metric*.
- The first equation of (1), called the *homogeneous Maxwell equation*, implies that $F \in \Omega^2(M)$ defines a cohomology class $[F] \in H^2_{\mathrm{dR}}(M)$. When this class is trivial, we can write $F = dA$ for some 1-form, called the *potential*. With this, the homogeneous Maxwell equation is automatically satisfied, and the second, *inhomogeneous* Maxwell equation in (1) becomes a PDE for the potential A controlled by the action functional

$$S(A) = \langle F(A), F(A) \rangle = \int_M F \wedge \star F.$$

- The representation of the field strength by the potential by means of the equation $F = dA$ is not unique: two potentials that differ by the derivative of a function $d\Lambda$ define the same field strength because $d^2 = 0$. The map $A \mapsto A + d\Lambda$ is called a *gauge transformation*. The relevant configuration space is therefore given by

$$\text{“Potentials/Gauge Transformations''} = \Omega^1(M)/d\Omega^0(M).$$

The transition to go from the field strength F to the potential is not completely trivial and requires some care. In one of the exercises of this week you will construct an exact sequence of the form

$$0 \longrightarrow H^1_{\mathrm{dR}}(M) \longrightarrow \Omega^1(M)/d\Omega^0(M) \xrightarrow{d} \Omega^2_{\mathrm{cl}}(M) \longrightarrow H^2_{\mathrm{dR}}(M) \longrightarrow 0,$$

where $\Omega^2_{\mathrm{cl}}(M)$ denotes the space of closed 2-forms. The way to think about this exact sequence is as follows: the *first* de Rham cohomology group controls the *injectively* of the map $d : \Omega^1(M)/d\Omega^0(M) \to \Omega^2_{\mathrm{cl}}(M)$, i.e., $H^1_{\mathrm{dR}}(M)$ is its *kernel*. On the other hand,

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the second cohomology group controls the *surjectivity* and $H^2_{\mathrm{dR}}(M)$ equals the *cokernel*. Let us discuss the effect of these groups:

i) When $H^1_{\mathrm{dR}}(M) \neq 0$, the configuration space $\Omega^1(M)/d\Omega^0(M)$ contains more than necessary to describe the field strength F, i.e., the electro-magnetic field. Therefore, at first, this space does not seem to be the correct one to describe the configuration space of Maxwell theory: it is "too big". This reasoning is correct when only Classical Mechanics is concerned, however for Quantum Mechanics the situation is different: The famous Aharanov–Bohm effect shows that Quantum Mechanical particles are "sensitive" to $H^1_{\mathrm{dR}}(M)$ and therefore the description in terms of the potential A, rather than that in terms of F is the correct one. (More details next lecture...)

ii) However, the equation $F = dA$ can only be solved for A when F is the trivial cohomology class in $H^2_{\mathrm{dR}}(M)$. When F represents a nontrivial class, we can only solve $F = dA$ *locally* in a neighborhood of any point, by Poincaré's Lemma. It was shown by Dirac, that working consistently with potentials A_U defined only locally over opens U, works Quantum Mechanically if the integrals

$$\int_{S^2} F \in 2\pi\sqrt{-1}\mathbb{Z}, \tag{2}$$

for any embedded 2-sphere $S^2 \subset M$. (Otherwise the Quantum-Mechanical wave function would not be single valued.) We have seen an example of such a configuration in the exercise about the magnetic monopole. Inserting the physical units, which involves putting the fundamental unit of electric charge in front of the integral above, this offers an explanation of the quantization of electric charge, by means of consistency of Quantum Mechanics.

Dirac's charge quantization condition actually has a beautiful topological interpretation using de Rham's theorem. Recall that the advantage of singular cohomology over de Rham cohomology is that for the former we can work over any abelian group, not necessarily $\mathbb{R}$ as in de Rham's cohomology group. For the singular chains, we can equally well work with $S^\infty_n(M,\mathbb{Z}) \subset S^\infty_n(M,\mathbb{R})$, i.e., finite sum of smooth singular chains with *integral* coefficients. The differential ∂ is obviously well-defined over the subgroup $S^\infty_n(M,\mathbb{Z}$ and this leads to the singular homology and cohomology with coefficients in $\mathbb{Z}$: $H_n^{\mathrm{sing}}(M,\mathbb{Z})$ and $H^n_{\mathrm{sing}}(M,\mathbb{Z})$. The inclusion $S^\infty_n(M,\mathbb{Z}) \subset S^\infty_n(M,\mathbb{R})$ leads to an obvious map $H^k_{\mathrm{sing}}(M,\mathbb{Z}) \to H^k_{\mathrm{sing}}(M,\mathbb{R})$.

Definition 1.1. We say that a closed k-form ω on M is *integral* if the image of the cohomology class $[\omega] \in H^k_{\mathrm{dR}}(M)$ is in the image of the map $H^k_{\mathrm{sing}}(M,\mathbb{Z}) \to H^k_{\mathrm{sing}}(M,\mathbb{R})$.

Unravelling the definition of de Rham's isomorphism this means that for an integral closed form $\omega \in \Omega^k(M)$,

$$\sum_i n_i \int_{\Delta^k} \sigma_i^* \omega \in \mathbb{Z},$$

for any closed smooth singular chain $\sigma := \sum_i n_i\sigma_i$, $n_i \in \mathbb{Z}$. In particular, taking the fundamental class of an embedded 2-sphere, this implies (2) for the 2-form $F/2\pi\sqrt{-1}$.

But what about the potential A? We can either work with *local* potentials, but this hides the geometry behind the mathematical theory of Maxwell's equations. To get a global picture, let us first recall some basics: we denote by $U(1)$ the circle group of complex numbers of unit length: $z = e^{i\theta}$ with multiplication

$$z_1 z_2 = e^{i(\theta_1+\theta_2)}.$$

We may either denote an element of $U(1)$ by z or θ. An *action* of $U(1)$ on a manifold X is given by a family $\varphi_z : X \to X$ of smooth diffeomorphisms of X subject to

$$\varphi_0 = \mathrm{id}_X, \quad \varphi_{z_1} \circ \varphi_{z_2} = \varphi_{z_1 z_2}.$$

An action of $U(1)$ on X induces a vector field ξ on X defined by

$$\xi(x) := \left.\frac{d}{dt}\varphi_{e^{it}}(x)\right|_{t=0},$$

called the *generating vector field* of the action. Finally, the action is called *free* if

$$\varphi_z(x) = x,\ \forall x \in X \quad \Longrightarrow \quad z = 1.$$

Theorem 1.2. *Given an integral closed 2-form $F/2\pi i$ on a manifold M there exists a triple (P, π, A) with:*

i) P is a smooth manifold equipped with a free action of $U(1)$,

ii) $\pi : P \to M$ is a surjective submersion for which $U(1)$ acts along the fibers of π, inducing an isomorphism $P/U(1) \cong M$. Furthermore, π is locally trivial *in the following sense: for each $x \in M$ there exists an open neighborhood U together with an isomorphism $\phi_U : \pi^{-1}(U) \xrightarrow{\cong} U \times U(1)$, compatible with the projection in the sense that the following diagram commutes:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi_U"] \arrow[d, "\pi"] & U \times U(1) \arrow[dl, "\mathrm{proj}"] \\ U & \end{tikzcd}$$

iii) A is a 1-form satisfying:
- *it is $U(1)$-invariant, i.e., $\varphi_z^* A = A$, $\forall z \in U(1)$,*
- *evaluating on the generating vector field gives $A(\xi) = 1$,*
- *$dA = \pi^F$.*

Proof. [1] We give the proof in the case that M is simply-connected. Choose a base-point $x_0 \in M$ and denote by $\mathcal{P}(x_0)$ the space of all piece-wise smooth path in M starting in x_0. When two paths $\gamma_0, \gamma_1 \in \mathcal{P}(x_0)$ have the same end-point they are homotopic

[1] The proof can be omitted; we also did not do this in the lecture.

by the assumption that M is simply connected, i.e., there is a piecewise smooth map $H : [0,1] \times [0,1] \to M$ with $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$ and $H(s,0) = x_0$, $H(s,1) = \gamma_0(1) = \gamma_1(1)$. In other words, there is an isomorphism

$$\mathcal{P}(x_0)/\sim_h \cong M$$

given by evaluating a path at its end-point. Now consider the product $\mathcal{P}(x_0) \times U(1)$. On this space we introduce an equivalence relation

$$(\gamma_0, z_0) \sim (\gamma_1, z_1) \quad \Longleftrightarrow \quad \gamma_0 \sim_h \gamma_1, \text{ and } z_0 = \exp\left(\int_H F\right) z_1.$$

Remark that this equivalence relation does not depend on the choice H of homotopy: any two choices define a (piecewise) smooth embedded 2-sphere and F has integral periods. It is easily checked that $\sim$ defines an equivalence relation and we define

$$P := (\mathcal{P}(x_0) \times U(1))/\sim .$$

The action

$$\varphi_z(\gamma, z') := (\gamma, zz')$$

of $U(1)$ on the product $\mathcal{P}(x_0) \times U(1)$, descends to the quotient and defines a smooth action of $U(1)$ on P. The obvious projection $\mathcal{P}(x_0) \times U(1) \to \mathcal{P}(x_0)$ induces a projection $\pi : P \to M$ compatible with the $U(1)$-action in the way stated in the theorem.

Let us now prove that the projection $\pi : P \to M$ is locally trivial. For this we fix a point $x_1 \in M$ and consider a small open neighborhood U of x_1 in M over which we can solve $F = dA_U$ for some 1-form $A_U \in \Omega^1(U)$. The inverse image $\pi^{-1}(U)$ consists of all equivalence classes $[(\gamma, z)]$ with $\gamma(1) \in U$. Let us choose a fixed path γ' from x_0 to x_1 so that we can represent each element in $\pi^{-1}(U)$ by a pair $[(\gamma, z)]$ where the path γ first follows γ' from x_0 to x_1. Let us now define a map $\phi_U : \pi^{-1}(U) \to U \times U(1)$ by

$$\phi_U([(\gamma, z)]) := (\gamma(1), \exp\left(\int_{x_1}^{\gamma(1)} A_U\right)$$

Stokes' theorem shows that this map is well-defined, i.e., independent of the chosen representative of the equivalence class. It is now easy to check that this defines a local trivialization of $\pi : P \to M$.

On another open subset V with basepoint x_2 we now gave another trivialization $\phi_V : \pi^{-1}V \to V \times U(1)$. On the overlap $U \cap V$ we therefore have

$$\phi_U \circ \phi_V^{-1}(x, z) = (x, \varphi_{UV}(x) z),$$

with the function $\varphi_{UV} : U \cap V \to U(1)$ given by

$$\varphi_{UV}(x) := \exp\left(\int_{x_1}^{x} A_U - \int_{x_2}^{x} A_V\right)$$

On $U \cap V$ we therefore have

$$A_U - A_V = d\varphi_{UV} \varphi_{UV}^{-1}. \tag{3}$$

Consider the 1-form $d\theta$ on $U(1)$. With this we define the following 1-form on $\pi^{-1}(U)$:

$$\pi^* A_U - \phi_U^* d\theta$$

A small computation using (3) then shows that on $\pi^{-1}(U) \cap \pi^{-1}(V)$ we have $A_U - \phi_U^* d\theta = A_V - \phi_V^* d\theta$ so the 1-forms glue together to form a global 1-form A satisfying $dA = \pi^* F$. □

We see that, at the price of going to a (slightly) bigger space P, we can now solve the equation $F = dA$. But how should we think of this space P? And what is the potential A? We will see in the next section that P is an example of a *principal bundle*, a notion that makes sense for any Lie group G. On this principal $U(1)$-bundle, the potential $A \in \Omega^1(P)$ can be interpreted as a *connection* with *curvature* F. In the coming lectures, we shall see that the integrality assumption on F is fits naturally into the theory of principal bundles because $F/2\pi i$ represents the *first Chern class* of P. Principal bundles for Lie groups other than $U(1)$ are important for the mathematical description of Yang–Mills theories, so we will develop the theory in general.

2. Principal bundles

The mathematical setting for gauge theories is provided by the theory of principal fiber bundles. Before we introduce these, we first consider some facts from the theory of Lie groups.

2.1. **Lie groups.** Here we give a list of the (minimal) facts that we need from the theory of Lie groups. At the end of this subsection we will illustrate all statements in the example of $SU(N)$. If the reader is not familiar with the theory of Lie groups, it will suffice to replace G by $SU(N)$ in the rest of the lecture notes.

- A Lie group G is a smooth manifold equipped with a group structure such that all group operations are smooth maps.
- The tangent space at the unit $T_eG = \mathfrak{g}$ is a *Lie algebra*: it is a vector space equipped with an antisymmetric bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity

  $$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0, \quad \text{for all } X,Y,Z \in \mathfrak{g}.$$

  When the group is abelian, i.e., $g_1g_2 = g_2g_1$ for all $g_1, g_2 \in G$, the Lie bracket is zero.
- There exists an *exponential map* $\exp : \mathfrak{g} \to G$ which is a diffeomorphism onto an open neighborhood of the unit. The exponential map does *not* satisfy

  $$e^X e^Y = e^{X+Y},$$

  unless it is abelian. There is a general power series expansion (the Campbell–Baker–Hausdorff formula) of the product on G in terms of iterated Lie brackets on $\mathfrak{g}$.

- The Lie group G acts on $\mathfrak{g}$ by the so-called *adjoint action*: $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$. In fact this action is linear and therefore a *representation* of G.

For $G = SU(N)$ this means the following: $SU(N)$ is the following group of matrices:

$$SU(N) := \{A \in \mathrm{Mat}_{N\times N}(\mathbb{C}),\ A^*A = 1,\ \det(A) = 1\}.$$

Its Lie algebra is given by

$$\mathfrak{su}(N) := \{X \in \mathrm{Mat}_{N\times N}(\mathbb{C}),\ A^* = -A,\ \mathrm{Tr}(A) = 0\},$$

equipped with the Lie bracket of matrices: $[X, Y] = XY - YX$. The exponential map is just given by the matrix exponential

$$e^X := \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

and we see that indeed $e^X \in SU(N)$ for $X \in \mathfrak{su}(N)$. The adjoint representation is just given by conjugation of matrices: $\mathrm{Ad}_A(X) = AXA^{-1}$.

A smooth action of a Lie group G on a manifold X is defined as for $G = U(1)$. For an element $g \in G$ we shall simply write $x \mapsto xg$ for the diffeomorphism defined by $g \in G$, and we have

$$xe = x \text{ for all } x \in X, \quad (xg)h = x(gh), \text{ for all } g, h \in G.$$

Given a smooth action, each $X \in \mathfrak{g}$ induces a *generating vector field* ξ_X defined by

$$\xi_X(x) := \left.\frac{d}{dt} xe^{tX}\right|_{t=0},$$

2.2. **Principal G-bundles.** Let M be a smooth manifold, and fix a Lie group G.

Definition 2.1. A *principal G-bundle* is a given by a surjective submersion $\pi : P \to M$, with a free right action of G on P along the fibers of π such that $P/G \cong M$, which is locally trivial in the following sense: for each $x \in M$ there exists an open neighborhood U together with an isomorphism $\phi_U : \pi^{-1}(U) \xrightarrow{\cong} U \times G$, compatible with the G-action and the projection in the sense that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\phi_U"] \arrow[d, "\pi"'] & U \times G \arrow[dl, "\mathrm{proj}"] \\
U &
\end{tikzcd}$$

Remark 2.2. Here is some standard terminology connected to principal bundles: G is called the *structure group* and P the *total space*. The map π is often referred to as the *projection* onto the *base* M. For a point $x \in M$, the inverse image $\pi^{-1}(x) \subset P$ is called the *fiber* over x, also denoted P_x. Remark that by definition, each fiber carries a free and *transitive* action of G.

Example 2.3. Let us list some examples of principal G-bundles.

i) Over any manifold M we always have the trivial G-bundle $M \times G \to M$, where G acts on itself by right translation.

ii) As an example of a *nontrivial* fiber bundle, consider the Hopf fibration $S^3 \to S^2$ with structure group $U(1)$, see the exercises.

iii) A rich source of examples of principal G bundles comes from the following theorem: if $G \subset G'$ is a closed compact subgroup of a Lie group, the map $\pi : G' \to G'/G$ is a principal G-bundle. (This means that one can prove that the bundle is locally trivial in this case.) As an example, consider the map $O(n+1) \to S^n$, a principal $O(n)$-bundle.

iv) The map $\pi : U(1) \to U(1)$ given by $z \mapsto z^n$ is a principal $\mathbb{Z}_n$-bundle.

A *morphism of principal G-bundles* is a smooth map $f : P_1 \to P_2$ which commutes with the right G-action. A section s of a principal G-bunde is a smooth map $s : X \to P$ satisfying $\pi \circ s =$identity.

As remarked above, we always have the trivial G-bundle $M \times G \to M$, but, as before we are also interested in nontrivial ones. Here is a useful criterion to decide wether a principal G-bundle is trivial or not:

Lemma 2.4. *A G-bundle P is isomorphic to the trivial G-bundle $M \times G$ if and only if it has a global section.*

Remark 2.5. There is a "cocycle view" on principal bundles over M as follows: by definition, we can find an open covering $\{U_\alpha\}_{\alpha \in I}$ of M such that P has local trivialisations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$. Two local trivializations (U_α, ϕ_α), and (U_β, ϕ_β) define a smooth map

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$

by the equation

$$(\phi_\alpha \circ \phi_\beta^{-1})(x, g) = (x, g\varphi_{\alpha\beta}(x)).$$

(Remark that for each x, $\varphi_{\alpha\beta}(x)$ us uniquely defined because G acts freely and transitively on the fibers of $\pi : P \to M$.) These functions are called *transistion functions*. One easily verifies the following conditions satisfied by the transition functions of a principal bundle:

i) for three local trivializations (U_α, h_α), (U_β, h_β) and (U_γ, h_γ),

$$\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = 1,$$

on $U_\alpha \cap U_\beta \cap U_\gamma$,

ii) for each local trivialization (U_α, h_α),

$$\varphi_{\alpha\alpha} = 1.$$

iii) for all pairs of local trivializations,

$$\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$$

Conversely, given an open covering $\{U_\alpha\}_{\alpha\in I}$ of M together with functions $\{\varphi_{\alpha,\beta} \in C^\infty(U_{\alpha\beta}, G)\}$ satisfying the two properties above, we can construct a principal bundle P over M as the quotient

$$P := \coprod_{\alpha\in I}(U_\alpha \times G) \Big/ \sim,$$

where

$$(x_\alpha, g_\alpha) \sim (x_\beta, g_\beta) \Longleftrightarrow x_\alpha = x_\beta \in U_\alpha \cap U_\beta,\ g_\alpha = g_\beta\varphi_{\alpha\beta}(x_\alpha).$$

From this point of view, a smooth section s of P is given by a collection $\{s_\alpha\}_{\alpha\in I}$ of smooth functions $s_\alpha : U_\alpha \to G$ satisfying $s_\beta\varphi_{\alpha\beta} = s_\alpha$.

Remark 2.6 (The gauge group)**.** Let $P \to M$ be a principal G-bundle. The set of all smooth morphisms from P to itself forms a group under composition[2], called the the *gauge group* $\mathcal{G}(P)$ in physics. To understand this group better, pick an element $\varphi \in \mathcal{G}(P)$. This element maps any $p \in P$ to another point $\varphi(p) \in \pi^{-1}(\pi(p))$ in the fiber over $\pi(p) \in M$, and therefore corresponds to the action of a *unique*[3] $g \in G$: $\varphi(p) = pg$. In this way, we obtain a (smooth) map $\tilde{\varphi} : P \to G$ satisfying

$$\tilde{\varphi}(pg) = g^{-1}\tilde{\varphi}(p)g, \quad \text{for all } g \in G.$$

In other words: this $\tilde{\varphi}$ is nothing but a smooth section of the fiber bundle $\mathrm{Ad}(P) := P \times_G G$ associated to the conjugation action $h \mapsto ghg^{-1}$ of G on itself. Remark that naievely both P and $\mathrm{Ad}(P)$ are fiber bundles with fiber "equal to G", but closer inspection show that for $\mathrm{Ad}(P)$ the fiber is *canonically* given by G, whereas for P the fiber is only non-canonically isomorphic to G, and therefore the section of P do not form a group. Only for the trivial g-bundle $P = M \times G$, –where the fibers are canonically isomorphic to G–, we have that $\mathrm{Ad}(P) = M \times G = P$. In view of this, elements $g_{\alpha\beta}$ of the cocycle of Remark 2.5 are called *local gauge transformations*.

3. Vector bundles

An important class of fiber bundles are given by vector bundles: these are fiber bundles with typical fiber a vector space V:

Definition 3.1. A *vector bundle of rank r* is given by a manifold E together with a smooth map $\pi : E \to M$ and the structure of an r-dimensional vector space on the fibers $E_x := \pi^{-1}(x)$ which is *locally trivial* in the following sense: each $x \in M$ has an open

[2] Check this! Why is any morphism automatically invertible?

[3] recall that the action of G is free and transitive along the fibers of π.

neighborhood U such that there exists a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{C}^r$ making the following diagram commutative

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times \mathbb{C}^r \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

and which is linear over each fiber. A *line bundle* is a vector bundle of rank one.

Remark 3.2. We can consider real or complex vector bundles, depending on whether the fibers are vector spaces over $\mathbb{R}$ or $\mathbb{C}$. With a view on applications in Quantum Mechanics, which always works in a complex Hilbert space, our main focus will be on complex vector bundles.

Example 3.3. Over any manifold M, we always have the trivial vector bundles $M \times \mathbb{R}^r$ (real case) and $M \times \mathbb{C}^r$ (complex). A general vector bundle E need not be of this form: although it is (by definition) *locally* a trivial vector bundle, globally it may not be trivial, but "twisted". The easiest example of a (real) twisted vector bundle is given by the Möbius line bundle over the circle: first we write $S^1 = \mathbb{R}/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}$ by translations $x \mapsto x + n$. We construct a line bundle L over S^1 by taking the quotient of the trivial line bundle over $\mathbb{R}$:

$$L := (\mathbb{R} \times \mathbb{R})/\mathbb{Z},$$

where we let $\mathbb{Z}$ act on $\mathbb{R} \times \mathbb{R}$ by either

$$(x, y) \mapsto (x + n, y), \quad \text{or} \quad (x, y) \mapsto (x + n, (-1)^n y).$$

In both cases, projection onto the first coordinate defines a smooth map $L \to S^1$ turning L into a line bundle over the circle (check!). In the first case we get the trivial line bundle $S^1 \times \mathbb{R}$, in the second case not: this is the Möbius line bundle as it "flips" as we go around the circle once.

Remark 3.4. There is a very concrete point of view on vector bundles using cocycles: Let $M = \bigcup_\alpha U_\alpha$ be a cover of M such that over each U_α there is a trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{C}^r$. (By definition, such a cover exists.) Over $U_\alpha \cap U_\beta$ we have two trivializations:

$$\begin{tikzcd} \pi^{-1}(U_{\alpha\beta}) \arrow[r, "\varphi_\alpha"] \arrow[d, "\varphi_\beta"'] & U_{\alpha\beta} \times \mathbb{C}^r \\ U_{\alpha\beta} \times \mathbb{C}^r \arrow[ur, "\varphi_\alpha \circ \varphi_\beta^{-1}"'] & \end{tikzcd}$$

Since both φ_α and φ_β are compatible with the projection to the base we can write

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, v) = (x, \varphi_{\alpha\beta}(v)), \quad \text{for } x \in U_{\alpha\beta}, v \in \mathbb{C}^r,$$

with $\varphi_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{C})$. We now shift our attention to these "transition functions" $\varphi_{\alpha\beta}$. The following properties are easily derived:

$$
\begin{aligned}
\varphi_{\alpha\alpha} &= 1 \\
\varphi_{\beta\alpha} &= \varphi_{\alpha\beta}^{-1} \\
\varphi_{\alpha\beta}\varphi_{\gamma\alpha}\varphi_{\beta\gamma} &= 1.
\end{aligned} \tag{4}
$$

These transition functions completely determine the vector bundle E: Given $\{\varphi_{\alpha\beta}\}_{\alpha,\beta\in I}$ satisfying the three properties above, define

$$E := \left(\coprod_{\alpha\in I} U_\alpha \times \mathbb{C}^r \right) / \sim$$

where

$$(x_\alpha, v) \sim (x_\beta, v) \Longleftrightarrow x_\alpha = x_\beta \in U_\alpha \cap U_\beta,\ \varphi_{\alpha\beta}(v) = w.$$

The properties satisfied by the $\varphi_{\alpha\beta}$ above guarantee that this defines an equivalence relation making the quotient well-defined.

Example 3.5 (The tangent bundle)**.** For any smooth manifold M, its tangent bundle is a real vector bundle: using a coordinate chart we can define local trivializations. The transition cocycles are given by the Jacobian matrices of the changes of coordinates. When the tangent bundle TM is trivial (or rather isomorphic to the trivial vector bundle of rank equal to the dimension n of M), we say that the M is parallelizable. This means that there exist n-vector fields $X_1, \ldots, X_n$ that at each point $x \in M$ form a basis of T_xM.

Example 3.6 (The universal line bundle over $\mathbb{P}^n$)**.** Recall the manifold structure of projectve space $\mathbb{P}^n$. Consider the following set:

$$T := \{(v, L) \in \mathbb{C}^{n+1} \times \mathbb{P}^n,\ v \in L\}.$$

There is an obvious projection $T \to \mathbb{P}^n$ projecting onto the second component. Clearly, the fiber $T_L,\ L \in \mathbb{P}^n$ is a vector space of dimension 1. In order to be a line bundle, we have to show local triviality. Over the domain $U_i := \{[z_0, \ldots, z_n],\ z_i \neq 0\}$ of the chart φ_i given in (**??**) there is an bijection

$$\varphi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{C},$$

given by the fact that any vector v in the line spanned by $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ can be written as

$$v = \lambda(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, 1, \frac{z_{i+1}}{z_i}, \ldots \frac{z_n}{z_i}), \quad \lambda \in \mathbb{C}.$$

(Dividing by z_i ensures that v determines λ uniquely.) The map above maps v to λ. This shows that $T \to \mathbb{P}^n$ is indeed a line bundle.

To determine the cocycle $\varphi_{ij} \in C^\infty(U_{ij}, \mathbb{C}^*)$[4] underlying this line bundle we consider the composition

$$U_{ij} \times \mathbb{C} \xrightarrow{\varphi_j^{-1}} \pi^{-1}(U_{ij}) \xrightarrow{\varphi_i} U_{ij} \times \mathbb{C},$$

which maps

$$([z_0, \ldots, z_n], \lambda) \mapsto ([z_0, \ldots, z_n], \frac{z_i}{z_j}\lambda).$$

The cocycle is therefore given by $\varphi_{ij}([z_0, \ldots, z_n]) = \frac{z_i}{z_j}$.

There is a natural way to get vector bundles from principal bundles:

Remark 3.7 (Pull-back of vector bundles)**.** Let $f : M \to N$ be a smooth map, and let $p : E \to N$ be a vector bundle over N. It is easy to see that

$$f^*E = \{(x, e) \in M \times E,\ f(x) = p(e)\}$$

has a canonical vector bundle structure over X.

Remark 3.8 (Linear algebra constructions with vector bundles)**.** Let E and F be vector bundles over M. It is not difficult to show that one can extend the standard constructions from linear algebra to define the following vector bundles over M:

i) the direct sum: $E \oplus F$,
ii) the tensor product $E \otimes F$,
iii) $\mathrm{Hom}(E, F) \cong E^* \otimes F$.

A section of a vector bundle $\pi : E \to X$ is a continuous map $s : M \to E$ such that

$$\pi \circ s = 1,$$

where the 1 on the right hand side means the constant function on X with that value. Denote the space of sections of E by $\Gamma(X, E)$. When E is smooth, one can require section to be be smooth maps as well, and this defines the space of smooth sections $\Gamma^\infty(X, E)$.

Principal bundles and vector bundles. Principal bundles are closely related to vector bundles. The main construction is the following way to get vector bundles from principal bundles:

Proposition 3.9 (Associated vector bundle)**.** *Let $\pi : P \to M$ be a principal G bundle and $\rho : G \to GL(V)$ a representation of G on a vector space V. The space*

$$P \times_G V := (P \times V) \,/\, \sim,$$

where

$$(p_1, v_1) \sim (p_2, v_2) \Longleftrightarrow \exists g \in G,\ (p_1, v_1) = (p_2 g, \rho(g) v_2),$$

has a canonical vector bundle structure.

[4]Recall that $GL(1, \mathbb{C}) = \mathbb{C}^*$

We can also go back: Let $E \to M$ be a vector bundle of rank r. Define a space

$$F(E) := \bigcup_{x \in M} F(E_x),$$

where $F(E_x)$ is the space of all bases in E_x: a point $e_x \in F(E_x)$ is a basis $e_x := (e_x^1, \ldots, e_x^r)$ of the vector space E_x. Using the local triviality of the vector bundle $E \to M$, one can define a smooth manifold structure on $F(E)$, such that the obvious projection $\pi : F(E) \to M$ turns $F(E)$ into a fiber bundle. There is an action of $GL(r, \mathbb{C})$ on $F(E)$ preserving the fibers which moves one basis to another. We conclude that $F(E)$ is a principal $GL(r, \mathbb{C})$-bundle.